STRUCTURE of the DIRAC-BORN-INFELD LAGRANGIANS
INCORPORATION OF $U(1)$ FIELDS

The Dirac-Born-Infeld Lagrangian is

$$\boxed{\mathcal{L} = \sqrt{\det |g_{ij} + F_{ij}|} = \sqrt{\det \left| \frac{\partial X^\mu}{\partial x_i} \frac{\partial X_\mu}{\partial x_j} + F_{ij} \right|}.}$$

Consider the case in 4 dimensions The expression under the square root can be expressed in terms of four sorts of terms, the first being

$$\boxed{\epsilon_{ijkl}\epsilon_{pqrs} g_{ip} g_{jq} g_{kr} g_{ls}}$$

This may be written as a sum of squares, the sum being over all permutations of μ, ν, ρ, σ;

$$\boxed{\Sigma \left(\epsilon_{ijkl} \frac{\partial X^\mu}{\partial x_i} \frac{\partial X^\nu}{\partial x_j} \frac{\partial X^\rho}{\partial x_k} \frac{\partial X^\sigma}{\partial x_l} \right)^2}$$

. Similarly the final term

$$\boxed{\epsilon_{ijkl}\epsilon_{pqrs} F_{ip} F_{jq} F_{kr} F_{ls} = \Sigma \left(\epsilon_{ijkl} F_{ij} F_{kl} \right)^2}$$

is a square in virtue of the fact that it is the determinant of an antisymmetric matrix. However, the cross terms

$$\boxed{\epsilon_{ijkl}\epsilon_{pqrs}g_{ip}g_{jq}F_{kr}F_{ls}}$$

are also sums of squares;

$$\boxed{\Sigma\left(\epsilon_{ijkl}\frac{\partial X^{\mu}}{\partial x_i}\frac{\partial X^{\nu}}{\partial x_j}F_{kl}\right)^2}$$

so the whole thing is quadratic. Moreover if a Clebsch parametrisation is adopted for the gauge potential, so that the gauge field F_{ij} has been replaced by a **Lagrange Bracket**, $\boxed{F_{ij} = [D_i,\ D_j],}$ where

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial X^{\mu}}{\partial x_i}\frac{\partial X_{\mu}}{\partial x_j} + \left(\frac{\partial p}{\partial x_i}\frac{\partial q}{\partial x_j} - \frac{\partial p^{\alpha}}{\partial x_j}\frac{\partial q^{\alpha}}{\partial x_i}\right)\right|},$$

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where

$$\boxed{D_i = \frac{\partial}{\partial x_i} + \sum_\alpha p^\alpha \frac{\partial q_\alpha}{\partial x_i}.}$$

- $\boxed{\mathcal{L} \text{ is reparametrisation invariant.}}$ Depending upon the number of $p, \& \ q$'s and the number of dimensions,

$F \wedge F = 0, \ F \wedge F \wedge F = 0$ etc. Then the expression under the square root is a quadratic form

$$\frac{1}{n!} \sum_{\text{permutations}} \left(\epsilon_{i_1, i_2, \ldots i_n} \frac{\partial X^{\mu_1}}{\partial x_{i_1}} \frac{\partial X^{\mu_2}}{\partial x_{i_2}} \cdots \frac{\partial X^{\mu_n}}{\partial x_{i_d}} \right)^2$$

two of the indices $\mu_1, \ \mu_2$, say X^{μ_1} may be identified with p_1^α and X^{μ_2} with q_1^α with the proviso that if one p appears so must the corresponding q In other words the general Brane Lagrangian is expressible as the square root of a sum of squares of Jacobians.

BRANES in the WRONG DIMENSIONS
LAGRANGIANS

Theoretical Physics makes progress often by extending the domain of validity of the fundamental equations which govern the system. Take, for example the Nambu-Goto String Lagrangian,

$$\boxed{\mathcal{L}_1 = \sqrt{\left[\left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\mu}{\partial \tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial \sigma}\right)^2\left(\frac{\partial X^\nu}{\partial \tau}\right)^2\right]}}$$

As a string theory, it describes the motion of a string, co-ordinatised by $X_\mu(\sigma, \tau)$ in d dimensions and parametrised by the base space coordinates σ, τ. Now consider interchanging the role of target and base space. The theory then becomes a field theory of two fields (ϕ, ψ) in d dimensions, with Lagrangian

$$\boxed{\mathcal{L}_2 = \sqrt{\left[\left(\frac{\partial \phi}{\partial x_\mu}\frac{\partial \psi}{\partial x_\mu}\right)^2 - \left(\frac{\partial \phi}{\partial x_\mu}\right)^2\left(\frac{\partial \psi}{\partial x_\nu}\right)^2\right]}}$$

another way to write this expression is as

$$\boxed{\mathcal{L}_2 = \sqrt{\frac{1}{2}\Sigma_{\mu,\ \nu}\left(\frac{\partial\phi}{\partial x_\mu}\frac{\partial\psi}{\partial x_\nu} - \frac{\partial\phi}{\partial x_\nu}\frac{\partial\psi}{\partial x_\mu}\right)^2} = \sqrt{\Sigma(J_{\mu\nu})^2}}$$

The expression under the square root is of the same structure as the Klein Gordon Lagrangian, with $J_{\mu\nu}$ playing the role of the gradient ϕ_μ

The motivation behind this idea is to model the transition between point particle mechanics and Klein Gordon field theory

In standard d dimensional Mechanics, the description of free particle motion by the classical point particle Lagrangian

$$\boxed{\mathcal{L}_1 = \sqrt{\Sigma\left(\frac{\partial X^\mu}{\partial\tau}\right)^2}}$$

goes over in terms of a field theory, to that given by the Lagrangian of a Klein Gordon field;

$$\boxed{\mathcal{L}_2 = \frac{1}{2}\Sigma\left(\frac{\partial\phi}{\partial x_\mu}\right)^2}$$

(for a massless field). This may be cited as an example of particle-wave duality. Is there a similar alternative description of strings and branes?

The idea is that strings described by the Nambu-Goto String Lagrangian $\mathcal{L}_1$ should analogously give rise to terms of two fields with Lagrangian which is some power of $\mathcal{L}_2$

Likewise, a simple brane Lagrangian, $\sqrt{\det\left|\frac{\partial X^\mu}{\partial\sigma_i}\frac{\partial X^\mu}{\partial\sigma_j}\right|}$ may be conjectured to be a field theory with as many components as there are world-volume co-ordinates with Lagrangian a power of

$$\begin{aligned}\mathcal{L} &= \det\left|\frac{\partial\phi^i}{\partial x_\mu}\frac{\partial\phi^j}{\partial x_\mu}\right| \\ &= \left(\frac{n!(d-n)!}{d!}\right)\Sigma\left(\frac{\partial\{\phi^1,\ \phi^2,\ldots,\phi^n\}}{\partial\{x_{\mu_1},x_{\mu_2}\ldots x_{\mu_n}\}}\right)^2.\end{aligned}$$

We call the second Lagrangian the

Companion Lagrangian

The original Lagrangian is diffeomorphism invariant; Reparametrisation Invariance, Diffeomorphism Invariance:

$$\boxed{\Sigma_a\, \phi^a_j \frac{\partial \mathcal{L}}{\partial \phi^a_k} = \delta_{jk}\mathcal{L}}$$

The Companion gives rise to covariant equations Covariance:

$$\boxed{\Sigma_j\, \phi^a_j \frac{\partial \mathcal{L}}{\partial \phi^b_j} = \delta_{ab}\mathcal{L}}$$

The Companion Lagrangian with square root is Covariant .

- It is homogeneous of weight one.

INTEGRABILITY

The Companion equations are integrable if the base space exceeds the target space by one. The Klein Gordon Lagrangian with a square root is

$$\boxed{\mathcal{L} = \sqrt{\Sigma \left(\frac{\partial \phi}{\partial x_\mu}\right)^2}}$$

In the minimal case of two base co-ordinates the equation of motion is the well known Bateman equation

$$\boxed{\left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial y^2} + \left(\frac{\partial \phi}{\partial y}\right)^2 \frac{\partial^2 \phi}{\partial x^2} - 2\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \phi}{\partial y}\right)\frac{\partial^2 \phi}{\partial x \partial y} = 0.}$$

The solution of the Bateman equation is given implicitly by solving for ϕ the equation

$$\boxed{xF(\phi(x,y)) + yG(\phi(x,y)) = 1,}$$

where $F,\ G$ are arbitrary functions

More Dimensions

The equations resulting from the square root of the Klein Gordon Lagrangian in more dimensions are just the sum of Bateman equations corresponding to the pairs of variables. A class of solutions is obtained from simultaneous solutions of these.

More Variables What is the situation with the next Lagrangian with 2 fields in 3 dimensions $(\mu, \nu = 1,\ 2,\ 3)$?

$$\boxed{\mathcal{L}_{\text{three}} = \sqrt{\Sigma \left[\left(\frac{\partial\phi}{\partial x_\mu} \frac{\partial\psi}{\partial x_\mu} \right)^2 - \left(\frac{\partial\phi}{\partial x_\mu} \right)^2 \left(\frac{\partial\psi}{\partial x_\nu} \right)^2 \right]}}$$

The solution to these equations is simply given by solving the following implicit equations for the unknowns $F_i(\phi, \psi), G_i(\phi, \psi)$

$$\boxed{x_1 F_1(\phi, \psi) + x_3 G_1(\phi, \psi) \;=\; 1}$$

$$\boxed{x_2 F_2(\phi, \psi) + x_3 G_2(\phi, \psi) \;=\; 1}$$

The generalisation should be evident.